Key

1a $f(x) = x, \quad -\pi \le x \le \pi$

$$b_k(f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(kx)\,dx$$

$$= \frac{1}{\pi}\left(x \cdot \frac{(-\cos(kx))}{k} \right)\Bigg|_{x=-\pi}^{\pi} - \underbrace{\frac{1}{\pi}\int_{-\pi}^{\pi} \frac{-\cos(kx)}{k}\,dx}_{=0}$$

$$= \frac{1}{\pi}\left(\frac{-\pi\cos(k\pi)}{k} + \frac{(-\pi)\cos(-k\pi)}{k} \right)$$

$$= \frac{1}{k}\left(-(-1)^k - (-1)^k \right)$$

$$= \frac{-2(-1)^k}{k}$$

$$f(x) = \sum_{k=1}^{\infty} \frac{-2(-1)^k}{k}\sin(kx)$$

$$f(\pi/2) = \pi/2$$

$$f(\pi/2) = \sum_{k=1}^{\infty} \frac{-2(-1)^k}{k}\underbrace{\sin(k\pi/2)}_{=\begin{cases} 0 & \text{if } k \text{ even} \\ 1 & \text{if } k = 1,5,9,13,\ldots \\ -1 & \text{if } k = 3,7,11,15,\ldots \end{cases}}$$

$(k = 2m+1)$

$$\Rightarrow \frac{\pi}{2} = \sum_{m=0}^{\infty} \frac{2}{2m+1}(-1)^m$$

$$= \frac{2}{1} - \frac{2}{3} + \frac{2}{5} - \frac{2}{7} + \cdots$$

1b (continued)

$g(0) = |0| = 0$

$$g(0) = \frac{\pi}{2} + \sum_{m=0}^{\infty} \frac{-4}{\pi(2m+1)^2} \cos(0)$$

$$\Rightarrow \frac{\pi^2}{8} = \sum_{m=0}^{\infty} \frac{1}{(2m+1)^2}$$

$$\pi = \sqrt{8\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots\right)}$$

1c How many terms to approximate π using each? (4 decimals)

$\pi = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7}$ Takes about 10,000 terms if you truncate, about 20,000 if you round.

$$\pi = \sqrt{8\left(1 + \frac{1}{9} + \frac{1}{25} + \dots\right)}$$

Takes about 8,000

Just give full credit for any answers between 3,000 and 50,000

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2 $h(x) = \begin{cases} x^2 & 0 \le x \le 1/2 \\ 0 & 1/2 \le x < 1 \end{cases}$

$$c_k(h) = \int_0^1 h(x)\, e^{-ik2\pi x}\, dx$$
$$= \int_0^{1/2} x^2 e^{-ik2\pi x}\, dx$$

$$c_0(h) = \int_0^{1/2} x^2\, dx$$
$$= (1/2)^3/3 = 1/24$$

$$c_k(h) = \int_0^{1/2} x^2 e^{-ik2\pi x}\, dx$$
$$= \frac{i(1-(-1)^k)}{4\pi^3 k^3} + \frac{(-1)^k}{4\pi^2 k^2} + \frac{i(-1)^k}{8\pi k}$$

$$h(x) = \frac{1}{24} + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \left(\frac{i(1-(-1)^k)}{4\pi^3 k^3} + \frac{(-1)^k}{4\pi^2 k^2} + \frac{i(-1)^k}{8\pi k} \right) e^{2\pi i k x}$$

Penalty if student didnt do separate calc for $k=0$, since this expression makes no sense at $k=0$

/8

3a $\|x^{1/4}\|_2 = \left(\int_0^1 |x^{-1/4}|^2\, dx\right)^{1/2}$

$$= \left(\int_0^1 x^{-1/2}\, dx\right)^{1/2}$$

$$= \sqrt{2}$$

3b $\|x^c\|_2 = \left(\int_0^1 |x^c|^2\, dx\right)^{1/2}$

$$= \left(\int_0^1 x^{2c}\, dx\right)^{1/2}$$

$\|x^c\|_2 < \infty$ exactly if $\int_0^1 x^{2c}\, dx < \infty$

$$\int_0^1 x^{2c}\, dx = \begin{cases} \left.\dfrac{x^{2c+1}}{2c+1}\right|_{x=0}^{1} & \text{if } c \neq -1/2 \\ \left.\operatorname{Log} x\right|_{x=0}^{1} & \text{if } c = -1/2 \end{cases}$$

Since $x^{2c+1}\big|_{x=0} < \infty$ for $c > -1/2$

$x^{2c+1}\big|_{x=0} = \infty$ for $c < -1/2$

$\operatorname{Log} 0 = \infty$,

we get

$\|x^c\|_2 < \infty$ exactly if $c > -1/2$.